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Limiting ε-subgradient characterizations of constrained best approximation

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Abstract

In this paper, we show that the strong conical hull intersection property (CHIP) completely characterizes the best approximation to any x in a Hilbert space X from the set

$$K := C \cap \{x \in X : -g(x) \in S\},$$

by a perturbation $x - l$ of x from the set C for some l in a convex cone of X, where C is a closed convex subset of X, S is a closed convex cone which does not necessarily have non-empty interior, Y is a Banach space and $g : X \to Y$ is a continuous S-convex function. The point l is chosen as the weak*-limit of a net of ε-subgradients. We also establish limiting dual conditions characterizing the best approximation to any x in a Hilbert space X from the set K without the strong CHIP. The ε-subdifferential calculus plays the key role in deriving the results.

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Keywords: Strong conical hull intersection property; ε-subgradients; Limiting dual conditions; Hilbert spaces; Constrained best approximation

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1. Introduction

When it comes to dual characterizations of constrained best approximation, it is not just the strong conical hull intersection property (CHIP) [6,10,14,17] that matters. The nature of the dual conditions is critical for a complete characterization. In this paper, we show that limiting dual conditions, which are described in terms of ε-subgradients, allow complete characterizations of constrained best approximation.

First, we investigate the problem of whether the best approximation to any x in a Hilbert space X from the set

$$K := C \cap \{x \in X : -g(x) \in S\},$$

can be characterized by the best approximation to a perturbation $x - l$ of x from the set C for some l in X, using the strong CHIP, where C is a closed convex subset of X, S is a closed convex cone in a Banach space Y, and $g : X \to Y$ is a continuous S-convex function. Solutions to this problem have recently been obtained in various cases of the set

$$\{x \in X : -g(x) \in S\}$$

in terms of the strong CHIP (see [6,10,14,17,18]). It is known that such a characterization of the so-called "perturbation property", in the nonlinear case of g, requires an additional regularity condition on g (see [14,17]), which is often restrictive in applications. In this paper we show that the strong CHIP completely characterizes the perturbation property without an additional regularity condition on g. We are able to achieve this by choosing l as the weak*-limit of a net of ε-subgradients [11,21,22]. We also obtain simple limiting dual conditions characterizing the best approximation from K under the strong CHIP.

Second, we examine whether a dual characterization of the best approximation to any x in a Hilbert space X from the set K holds in the absence of the strong CHIP. It is also known that dual characterizations of the best approximation from K in terms of subgradients require a constraint qualification (see [6,9,20,23]). We show that limiting dual conditions, which are described in terms of ε-subgradients, hold without any constraint qualifications. We give a numerical example to illustrate the nature of the limiting ε-subgradient conditions.

The layout of the paper is as follows. In Section 2 we collect definitions, notations and preliminary results that will be used later in the paper. In Section 3, we establish conditions for the perturbation property and other dual conditions under the strong CHIP. In Section 4, we present asymptotic dual conditions characterizing the best approximation in terms of ε-subgradients without the strong CHIP.

2. Preliminaries

We begin this section by fixing the notations, definitions and preliminaries that will be used later in the paper. Let X and Y be Banach spaces. The continuous dual space of X will be denoted by X^*. For a set $W \subset X^*$, the weak*-closure (resp. closure) of W will be denoted by $w^*\text{-cl}\, W$ (resp. $\text{cl}\, W$). We shall denote by $\text{int}\, A$ the interior of A, where A is a subset of X. For the subset A of X, the indicator function δ_A is defined by $\delta_A(x) = 0$ if $x \in A$

and $\delta_A(x) = +\infty$ if $x \notin A$. The support function σ_A is defined by

$$\sigma_A(x^*) = \sup_{x \in A} x^*(x) \quad (x^* \in X^*).$$

The epigraph of $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, Epi f, is defined by

$$\text{Epi } f = \{(x, r) \in X \times \mathbb{R} : x \in \text{dom } f,\ f(x) \leqslant r\},$$

where the domain of f, dom f, is given by

$$\text{dom } f = \{x \in X : f(x) < +\infty\}.$$

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then the conjugate function of f, denoted by $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, is given by

$$f^*(x^*) = \sup\{x^*(x) - f(x) \ : \ x \in \text{dom } f\}, \quad (x^* \in X^*).$$

Note that we have $\delta_D^* = \sigma_D$ for each subset D of X.

For $\varepsilon > 0$, the ε-subdifferential of f at $a \in \text{dom } f$ is defined as the non-empty weak* closed convex set

$$\partial_\varepsilon f(a) = \{x^* \in X^* : f(x) - f(a) \geqslant x^*(x - a) - \varepsilon \ \forall x \in \text{dom } f\}.$$

The elements of $\partial_\varepsilon f(a)$ are called ε-subgradients of f at a. For $\varepsilon = 0$, $\partial_0 f(a)$ is the usual subdifferential of f at a, and is often denoted by $\partial f(a)$. See Zalinescu [23] for a detailed discussion of this set and its properties. Note that $\bigcap_{\varepsilon > 0} \partial_\varepsilon f(a) = \partial f(a)$. It follows from the definitions of Epi f^* and the ε-subdifferential of f that if $a \in \text{dom } f$, then

$$\text{Epi } f^* = \bigcup_{\varepsilon \geqslant 0} \{(x^*, x^*(a) + \varepsilon - f(a)) \ : x^* \in \partial_\varepsilon f(a)\}.$$

For details see [12,13]. For convenience, we denote the composite mapping $\lambda \circ g$ by λg, where $\lambda \in Y^*$ and $g : X \longrightarrow Y$ is a function. For a subset W of X, define the negative dual cone of W by

$$W^{\ominus} := \{x^* \in X^* : x^*(w) \leqslant 0 \quad \forall\, w \in W\}$$

and the positive dual cone of W by $W^{\oplus} := -W^{\ominus}$. Also, $W^{\ominus}$ is called the normal cone of W at 0. For the non-empty subset W of X, the conical hull of W is denoted by cone W. A function $g : X \longrightarrow Y$ is called S-convex if

$$\lambda g(x) + (1 - \lambda) g(y) - g(\lambda x + (1 - \lambda) y) \in S \quad (x, y \in X;\ 0 \leqslant \lambda \leqslant 1),$$

where S is a closed convex cone in Y. In particular, if $Y = \mathbb{R}$ and $S = \mathbb{R}_+ := \{r \in \mathbb{R} : r \geqslant 0\}$, then S-convex reduces to the usual definition of a convex function. Let C be a non-empty closed convex subset of X and let

$$D := \{x \in X : -g(x) \in S\}, \tag{2.1}$$

where $g : X \longrightarrow Y$ is a continuous S-convex function. It is easy to check that D is a closed convex subset of X. Let $K := C \cap D \neq \emptyset$. If K is non-empty, then using the Hanh–Banach separation theorem we obtain

$$\mathrm{Epi}\,\sigma_K = w^* - \mathrm{cl}(\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^* + \mathrm{Epi}\,\sigma_C). \tag{2.2}$$

Moreover, if D is non-empty then

$$\mathrm{Epi}\,\sigma_D = w^* - \mathrm{cl}(\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*). \tag{2.3}$$

A proof of this result is given in [2,13]. For a non-empty subset W of X and $x \in X$, we define the distance from x to W by

$$d(x, W) := \inf_{w \in W} \|x - w\|.$$

We recall (see [20]) that a point $w_0 \in W$ is called a best approximation for $x \in X$ (i.e. $w_0 \in P_W(x)$), if

$$d(x, W) = \|x - w_0\|.$$

If for each $x \in X$ there exists a unique best approximation $w_0 \in W$, then W is called a Chebyshev subset of X. Recall (see [7]) that every closed convex set in a Hilbert space is Chebyshev.

The following lemma, which gives a characterization of the best approximation in Banach spaces, is well known and is due, independently, to Deutsch [5] and Rubinstein [19] (see, e.g., [20,23]).

Lemma 2.1. *Let X be a Banach space, W be a closed convex subset of X, $x \in X$, and $w_0 \in W$. Then $w_0 \in P_W(x)$ if and only if there exists $f \in (W - w_0)^{\ominus}$ such that $\|f\| = 1$ and $f(x - w_0) = \|x - w_0\|$.*

3. Asymptotic perturbation properties

In this section, we assume that X and Y are Banach spaces and show that the strong CHIP characterizes an asymptotic perturbation property. We begin by recalling the notion of the strong CHIP, which was first defined for any finite collection of convex sets in a Hilbert space in [9], and which plays a central role for instance in constrained best approximation and optimization (see, e.g. [1,2,6,8–10]).

Definition 3.1. Let C_1, and C_2 be two closed convex sets in X and let $x \in C_1 \cap C_2$. Then $\{C_1, C_2\}$ is said to have the strong CHIP at x, if

$$(C_1 \cap C_2 - x)^{\ominus} = (C_1 - x)^{\ominus} + (C_2 - x)^{\ominus}.$$

The pair $\{C_1, C_2\}$ is said to have the strong CHIP if it has the strong CHIP at each $x \in C_1 \cap C_2$.

Note that if $C_1 \cap C_2 \neq \emptyset$, then we always have

$$(C_1 - x)^{\ominus} + (C_2 - x)^{\ominus} \subset (C_1 \cap C_2 - x)^{\ominus}, \quad (x \in X). \tag{3.1}$$

If $\operatorname{Epi} \sigma_{C_1} + \operatorname{Epi} \sigma_{C_2}$ is weak*-closed then the collection $\{C_1, C_2\}$ has the strong CHIP. For details see [2,4,3,14]. For each $x \in X$, define

$$\widetilde{M}(x) := \left\{x^* \in X^* \ : \ (x^*, x^*(x)) \in w^* - \operatorname{cl}(\cup_{\lambda \in S^{\oplus}} \operatorname{Epi}(\lambda g)^*)\right\}. \tag{3.2}$$

Clearly, $\widetilde{M}(x)$ is a convex cone in X^* as $(\cup_{\lambda \in S^{\oplus}} \operatorname{Epi}(\lambda g)^*)$ is a convex cone.

Proposition 3.1. *For each $x \in X$, $(D - x)^{\ominus} = \widetilde{M}(x)$, where D is the closed convex set defined by* (2.1).

Proof. The point $x^* \in (D-x)^{\ominus}$ if and only if $\sigma_D(x^*) \leqslant x^*(x)$, which, in turn, is equivalent to $(x^*, x^*(x)) \in \operatorname{Epi} \sigma_D$. Since

$$\operatorname{Epi} \sigma_D = w^* - \operatorname{cl}(\cup_{\lambda \in S^{\oplus}} \operatorname{Epi}(\lambda g)^*),$$

it follows that $x^* \in (D - x)^{\ominus}$ if and only if $(x^*, x^*(x)) \in w^* - \operatorname{cl}(\cup_{\lambda \in S^{\oplus}} \operatorname{Epi}(\lambda g)^*)$, which, by definition, is equivalent to the condition that $x^* \in \widetilde{M}(x)$. $\square$

It is worth noting that $\widetilde{M}(x)$ is a weak*-closed convex cone of X^*. We will now see how $\widetilde{M}(x)$ can be expressed in terms of ε-subgradients at x for each $x \in D$.

Lemma 3.1. *If $x \in D$, then*

$$\widetilde{M}(x) = \left\{ \begin{array}{c} x^* \in X^* : x^* = w^* - \lim\limits_{\alpha} x^*_\alpha, \ x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x), \ \{\varepsilon_\alpha\} \subset \mathbb{R}_+, \\ \{\lambda_\alpha\} \subset S^{\oplus}, \ \lim\limits_{\alpha} (\lambda_\alpha g)(x) = 0, \ \lim\limits_{\alpha} \varepsilon_\alpha = 0. \end{array} \right\}.$$

Proof. Let $x^* \in \widetilde{M}(x)$. Then, by definition (see (3.2)),

$$(x^*, x^*(x)) \in w^* - \operatorname{cl}(\cup_{\lambda \in S^{\oplus}} \operatorname{Epi}(\lambda g)^*). \tag{3.3}$$

On the other hand, we have

$$\operatorname{Epi}(\lambda g)^* = \cup_{\varepsilon \geqslant 0}\{(y^*, \varepsilon + y^*(x) - (\lambda g)(x)) : y^* \in \partial_\varepsilon(\lambda g)(x)\}.$$

So,

$$\begin{aligned} &(x^*, x^*(x)) \in w^* - \operatorname{cl}(\cup_{\lambda \in S^{\oplus}} \cup_{\varepsilon \geqslant 0} \{(y^*, \varepsilon + y^*(x) \\ &\quad -(\lambda g)(x)) : y^* \in \partial_\varepsilon(\lambda g)(x)\}). \end{aligned} \tag{3.4}$$

Now, we can find nets $\{\varepsilon_\alpha\}, \{r_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^{\oplus}$ and $\{x^*_\alpha\} \subset X^*$ with $x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$, for all α, such that

$$(x^*, x^*(x)) = w^* - \lim_{\alpha} (x^*_\alpha, \varepsilon_\alpha + x^*_\alpha(x) - (\lambda_\alpha g)(x)).$$

Thus,

$$x^* = w^* - \lim_{\alpha} x^*_\alpha \tag{3.5}$$

and

$$x^*(x) = \lim_{\alpha} [\varepsilon_\alpha + x_\alpha^*(x) - (\lambda_\alpha g)(x)]. \tag{3.6}$$

In view of (3.5) and (3.6), we obtain

$$\lim_{\alpha} x_\alpha^*(x) = x^\alpha(x) = \lim_{\alpha} [\varepsilon_\alpha + x_\alpha^*(x) - (\lambda_\alpha g)(x)],$$

which implies that

$$\lim_{\alpha} [\varepsilon_\alpha - (\lambda_\alpha g)(x)] = 0. \tag{3.7}$$

Since $x \in D$, $-g(x) \in S$, and so, $-(\lambda_\alpha g)(x) \geqslant 0$ for all α. This, together with (3.7) and the fact that $\varepsilon_\alpha \geqslant 0$ for all α, implies that $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = 0$.

Conversely, suppose that there exist nets $\{\varepsilon_\alpha\}, \{r_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^\oplus$ and $\{x_\alpha^*\} \subset X^*$ with $x_\alpha^* \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ such that $x^* = w^* - \lim_\alpha x_\alpha^*$, and $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = 0$. Since $x_\alpha^* \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ for all α, it follows that

$$(\lambda_\alpha g)(y) - (\lambda_\alpha g)(x) \geqslant x_\alpha^*(y - x) - \varepsilon_\alpha \quad \forall\, y \in X,\ \forall\, \alpha. \tag{3.8}$$

If $y \in D$, then $-g(y) \in S$, and so, $(\lambda_\alpha g)(y) \leqslant 0$ for all α. It now follows from (3.8) that

$$(\lambda_\alpha g)(x) \leqslant x_\alpha^*(x - y) + \varepsilon_\alpha \quad \forall\, y \in D\ \forall\, \alpha. \tag{3.9}$$

Since $x^*(y) = \lim_\alpha x_\alpha^*(y)$ for each $y \in X$, and $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = 0$, we obtain from (3.9) that

$$x^*(y - x) \leqslant 0 \quad \forall\, y \in D.$$

Hence, $x^* \in (D - x)^\ominus = \widetilde{M}(x)$, by Proposition 3.1. □

Let us now define a convex cone that is related to $\widetilde{M}(x)$. For each $x \in X$ and $\lambda \in S^\oplus$, define

$$C_\lambda(x) := \operatorname{cone}\{\partial(\lambda g)(x) : (\lambda g)(x) = 0\}$$

and

$$M(x) := \cup_{\lambda \in S^\oplus} C_\lambda(x).$$

It is easy to verify that $M(x)$ is a convex cone in X^* with $M(x) \subset \widetilde{M}(x)$. The following simple example illustrates that, in general, $M(x) \neq \widetilde{M}(x)$.

Example 3.1. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$g(x, y) = (x^2 + y^2)^{\frac{1}{2}} - y,$$

and let $x = (0, 1) \in \mathbb{R}^2$ and $S = \mathbb{R}_+$. Then for each $\lambda \in S^\oplus$, we have that $\lambda g(x) = 0$, $\partial(\lambda g)(x) = \{(0, 0)\}$, and so, $M(x) = \{(0, 0)\}$, whereas $\widetilde{M}(x) = (D - x)^\ominus = -(\mathbb{R} \times \mathbb{R}_+)$.

Moreover, a direct calculation shows that $(-1, 0, 0) \in \mathrm{cl}(\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*)$, but $(-1, 0, 0) \notin \cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$, and so $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is not closed.

We will now show that if $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed, then $M(x) = \widetilde{M}(x)$.

Proposition 3.2. *If the set $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed in $X^* \times \mathbb{R}$, then, for each $x \in D$, $M(x) = \widetilde{M}(x) = (D - x)^{\ominus}$.*

Proof. Let $x \in D$ be arbitrary. Then, the conclusion will follow from Proposition 3.1 if we show that $\widetilde{M}(x) \subset M(x)$. To see this, let $x^* \in \widetilde{M}(x)$. Since $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed, it follows from the definition of $\widetilde{M}(x)$ that $(x^*, x^*(x)) \in \cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$. So, there exists $\lambda \in S^{\oplus}$ such that for each $y \in D$, $x^*(y) - (\lambda g)(y) \leqslant x^*(x)$. If $x \in D$, then $-(\lambda g)(x) \leqslant 0$. This gives us $(\lambda g)(x) = 0$. Hence, for each $y \in D$, $x^*(y) - x^*(x) \leqslant (\lambda g)(y) - (\lambda g)(x)$, thus, $x^* \in \partial(\lambda g)(x)$, and hence, $x^* \in M(x)$. $\square$

Note that $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed if, in particular, the interior of S, $\mathrm{int}(S)$, is non-empty and $-g(x_0) \in \mathrm{int}(S)$ for some $x_0 \in X$. For details see [15], and for other generalized interior-point conditions see [16].

In the sequel, we assume that X is a Hilbert space. The following characterization of best approximation in Hilbert spaces is well known (see [7]).

Lemma 3.2. *Let X be a Hilbert space; and let W be a closed convex subset of X, $x \in X$, and $w_0 \in W$. Then $w_0 = P_W(x)$ if and only if $x - w_0 \in (W - w_0)^{\ominus}$.*

We will now show that the strong CHIP of $\{C, D\}$ completely characterizes a perturbation property. Let us first see a basic perturbation property of the best approximation.

Proposition 3.3. *Let $x \in X$, and let $x_0 \in K$. If $x_0 = P_C(x - l)$ for some $l \in \widetilde{M}(x_0)$. Then $x_0 = P_K(x)$.*

Proof. Assume that $x_0 = P_C(x - l)$ for some $l \in \widetilde{M}(x_0)$. In view of Lemma 3.2, we have $x - l - x_0 \in (C - x_0)^{\ominus}$, and so by (3.1) and Proposition 3.1, we get

$$\begin{aligned} x - x_0 &\in (C - x_0)^{\ominus} + l \\ &\subset (C - x_0)^{\ominus} + \widetilde{M}(x_0) \\ &= (C - x_0)^{\ominus} + (D - x_0)^{\ominus} \\ &\subset (K - x_0)^{\ominus}. \end{aligned}$$

Hence, $x - x_0 \in (K - x_0)^{\ominus}$. Again, applying Lemma 3.2, we have $x_0 = P_K(x)$. $\square$

Theorem 3.1. *Let $x_0 \in K$. Then the following assertions are equivalent:*

(1) *$\{C, D\}$ has the strong CHIP at x_0;*
(2) *For any $x \in X$,*

$$x_0 = P_K(x) \quad \textit{if and only if} \quad x_0 = P_C(x - l) \qquad \textit{for some} \quad l \in \widetilde{M}(x_0).$$

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. For any $x \in X$, by Proposition 3.3, we have if $x_0 = P_C(x-l)$ for some $l \in \widetilde{M}(x_0)$, then $x_0 = P_K(x)$. Therefore we only need to show that the converse is true. Suppose that $x_0 = P_K(x)$. By Lemma 3.2, we get $x - x_0 \in (K - x_0)^{\ominus}$. Using the strong CHIP, we can find $l_1 \in (C - x_0)^{\ominus}$ such that $x - x_0 - l_1 \in (D - x_0)^{\ominus}$. Since $(D - x_0)^{\ominus} = \widetilde{M}(x_0)$, it follows that $l := x - x_0 - l_1 \in \widetilde{M}(x_0)$. So, $x - l - x_0 = l_1 \in (C - x_0)^{\ominus}$. It now follows from Lemma 3.2 that $x_0 = P_C(x - l)$, and hence (2) holds.

(2) $\Longrightarrow$ (1). Assume that (2) holds. Let $z \in (K - x_0)^{\ominus}$ be arbitrary. Let $x = z + x_0 \in X$. Then, $x - x_0 = z \in (K - x_0)^{\ominus}$, and so by Lemma 3.2, we have $x_0 = P_K(x)$. Now, it follows from the assumption that there exists $l \in \widetilde{M}(x_0)$ such that $x_0 = P_C(x - l)$. Again by Lemma 3.2, we get $x - l - x_0 \in (C - x_0)^{\ominus}$. Therefore,

$$z = x - x_0 = x - l - x_0 + l \in (C - x_0)^{\ominus} + \widetilde{M}(x_0) = (C - x_0)^{\ominus} + (D - x_0)^{\ominus}.$$

This give us that

$$(K - x_0)^{\ominus} \subset (C - x_0)^{\ominus} + (D - x_0)^{\ominus}.$$

This, together with (3.1) implies that

$$(K - x_0)^{\ominus} = (C - x_0)^{\ominus} + (D - x_0)^{\ominus}.$$

Hence, $\{C, D\}$ has the strong CHIP at x_0. □

The following corollary follows easily from Theorem 3.1.

Corollary 3.1. *The following statements are equivalent*:

(1) $\{C, D\}$ *has the strong CHIP*;
(2) *For each* $x \in X$, $P_K(x) = P_C(x - l)$ *for some* $l \in \widetilde{M}(P_K(x))$.

Theorem 3.2. *Suppose that* $\{C, D\}$ *has the strong CHIP. Then for each* $x \in X$, *the element* $x_0 = P_K(x) \in K$ *satisfies* $P_K(x) = P_C(x - l)$ *for some* $l \in \widetilde{M}(x_0)$.

Proof. Let $x \in X$ be arbitrary. Since K is a Chebyshev set in X, we conclude that there exists $x_0 \in K$ such that $x_0 = P_K(x)$. Now, by Theorem 3.1, there exists $l \in \widetilde{M}(x_0)$ such that $x_0 = P_C(x - l)$. □

As a consequence of Theorem 3.2, we observe that if Epi $\sigma_C +$Epi σ_D is closed in $X^* \times \mathbb{R}$, then for each $x \in X$, the element $x_0 := P_K(x) \in K$ satisfies $P_K(x) = P_C(x - l)$ for some $l \in \widetilde{M}(x_0)$.

In the following we shall give an asymptotic dual characterization of best approximation using the strong CHIP. Recall that $\partial\|x - x_0\|(x_0)$ is given by

$$\partial\|x - x_0\|(x_0) = \{v \in X^* : \|v\| = 1,\ v(x - x_0) = \|x - x_0\|\}.$$

Theorem 3.3. *Let X be a Hilbert space; let $x \in X$ and $x_0 \in K$. Assume that $\{C, D\}$ has the strong CHIP at x_0. Then the following assertions are equivalent*:

(1) $x_0 = P_K(x)$;
(2) $0 \in \partial\|x - x_0\|(x_0) + (C - x_0)^{\ominus} + \widetilde{M}(x_0)$;
(3) $x_0 = P_C(x - l)$ *for some* $l \in \widetilde{M}(x_0)$.

Proof. [(1) $\Longleftrightarrow$ (2)]. Suppose that $x_0 = P_K(x)$. Then, it follows from Lemma 2.1 that there exists $v \in X^*$ such that $v \in (K - x_0)^{\ominus}$, $\|v\| = 1$, and $v(x - x_0) = \|x - x_0\|$. This implies that there exists $v \in X^*$ such that $-v \in \partial\|x - x_0\|(x_0)$ and $v \in (K - x_0)^{\ominus}$. Since $\{C, D\}$ has the strong CHIP at x_0, $(K - x_0)^{\ominus} = (C - x_0)^{\ominus} + (D - x_0)^{\ominus}$. Then for $v \in (K - x_0)^{\ominus}$ there exist $v_1 \in (C - x_0)^{\ominus}$ and $v_2 \in (D - x_0)^{\ominus}$ such that $v = v_1 + v_2$. Since $v_2 \in (D - x_0)^{\ominus} = \widetilde{M}(x_0)$, we conclude that $v \in (C - x_0)^{\ominus} + \widetilde{M}(x_0)$. But, we have $-v \in \partial\|x - x_0\|(x_0)$. Hence,

$$0 \in \partial\|x - x_0\|(x_0) + (C - x_0)^{\ominus} + \widetilde{M}(x_0).$$

Conversely, assume that

$$0 \in \partial\|x - x_0\|(x_0) + (C - x_0)^{\ominus} + \widetilde{M}(x_0).$$

Then there exist $v_1 \in \partial\|x - x_0\|(x_0)$, $v_2 \in (C - x_0)^{\ominus} + \widetilde{M}(x_0)$ such that $v_1 + v_2 = 0$. Since $\widetilde{M}(x_0) = (D - x_0)^{\ominus}$, we get $v_2 \in (C - x_0)^{\ominus} + (D - x_0)^{\ominus} = (K - x_0)^{\ominus}$.

Now, let $y \in K := C \cap D$ be arbitrary. Then,

$$v_2(y - x_0) \leqslant 0 \quad \forall\, y \in K. \tag{3.10}$$

Since $v_1 \in \partial\|x_0 - x\|(x_0)$ and $v_1 = -v_2$, it follows from (3.10) that

$$\|x_0 - x\| \leqslant \|x - y\| \quad \forall\, y \in K$$

and so

$$\|x - x_0\| \leqslant \inf_{y \in K} \|x - y\| = d(x, K) \leqslant \|x - x_0\|.$$

Hence, $\|x - x_0\| = d(x, K)$. That is, $x_0 = P_K(x)$.

[(3) $\Longleftrightarrow$ (1)]. The implication (3) $\Longrightarrow$ (1) follows from Proposition 3.3. To establish [(1) $\Longrightarrow$ (3)], suppose that (1) holds. Since $\{C, D\}$ has the strong CHIP at x_0 and $x_0 = P_K(x)$, it follows from Theorem 3.1 that $x_0 = P_C(x - l)$ for some $l \in \widetilde{M}(x_0)$. $\square$

Observe that the equivalence (1) $\Longleftrightarrow$ (2) holds in general Banach space X. The following theorem extends the corresponding results in [14,17].

Theorem 3.4. *Let X be a Hilbert space, and let $x_0 \in K$. The following assertions are equivalent*:

(1) $(K - x_0)^{\ominus} = (C - x_0)^{\ominus} + M(x_0)$;
(2) *For any* $x \in X$,

$$x_0 = P_K(x) \quad \textit{if and only if} \quad x_0 = P_C(x - l) \qquad \textit{for some} \quad l \in M(x_0).$$

Proof. (1) $\Longrightarrow$ (2). Assume that $x_0 = P_C(x - l)$ for some $l \in M(x_0)$. By Lemma 3.2, we have $x - l - x_0 \in (C - x_0)^{\ominus}$, and so

$$x - x_0 \in (C - x_0)^{\ominus} + l \subset (C - x_0)^{\ominus} + M(x_0) = (K - x_0)^{\ominus}.$$

Thus, $x - x_0 \in (K - x_0)^{\ominus}$. Again, applying Lemma 3.2, we have $x_0 = P_K(x)$. To see the converse implication, let $x \in X$, and let $x_0 = P_K(x)$. Then by Lemma 3.2, we get $x - x_0 \in (K - x_0)^{\ominus}$. Using (1), we can find $l \in M(x_0)$ such that $x - x_0 - l \in (C - x_0)^{\ominus}$. By Lemma 3.2, $x_0 = P_C(x - l)$, and hence (2) holds.
(2) $\Longrightarrow$ (1). Assume that (2) holds. Let $z \in (K - x_0)^{\ominus}$ be arbitrary. Let $x = z + x_0 \in X$. Then, $x - x_0 = z \in (K - x_0)^{\ominus}$, and so by Lemma 3.2, we have $x_0 = P_K(x)$. Now, it follows from the assumption that there exists $l \in M(x_0)$ such that $x_0 = P_C(x - l)$. Again by Lemma 3.2, we get $x - l - x_0 \in (C - x_0)^{\ominus}$. Therefore,

$$z = x - x_0 = x - l - x_0 + l \in (C - x_0)^{\ominus} + M(x_0).$$

This give us that

$$(K - x_0)^{\ominus} \subset (C - x_0)^{\ominus} + M(x_0).$$

On the other hand, as $M(x_0) \subset (D - x_0)^{\ominus}$, we have

$$(C - x_0)^{\ominus} + M(x_0) \subset (C - x_0)^{\ominus} + (D - x_0)^{\ominus} \subset (K - x_0)^{\ominus}.$$

Hence, $(K - x_0)^{\ominus} = (C - x_0)^{\ominus} + M(x_0)$. $\square$

In passing, observe that the condition (1) of Theorem 3.4 was called the Basic constraint qualification in [17], where $Y = \mathbb{R}^m$ and $S = \mathbb{R}^m_+$. Observe also from Proposition 3.2 that if $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed in $X^* \times \mathbb{R}$, then the condition $(K - x_0)^{\ominus} = (C - x_0)^{\ominus} + M(x_0)$ is equivalent to the strong CHIP.

We now deduce non-asymptotic dual characterizations of best approximations from Theorem 3.3 under the additional condition that $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed.

Corollary 3.2. *Let X be a Hilbert space; let $x \in X$ and $x_0 \in K$. Assume that $\{C, D\}$ has the strong CHIP at x_0 and that $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is weak*-closed in $X^* \times \mathbb{R}$. Then the following assertions are equivalent*:

(1) $x_0 = P_K(x)$;
(2) $0 \in \partial\|x - x_0\|(x_0) + \partial(\lambda g)(x_0) + (C - x_0)^{\ominus}$, *and* $(\lambda g)(x_0) = 0$ *for some* $\lambda \in S^{\oplus}$;
(3) $0 \in \partial\|x - x_0\|(x_0) + (C - x_0)^{\ominus} + M(x_0)$;
(4) $x_0 = P_C(x - l)$ *for some* $l \in M(x_0)$.

Proof. The implications (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (1) follow from Theorem 3.3, since $\widetilde{M}(x_0) = M(x_0)$. The implication (2) $\Longrightarrow$ (3) is obvious, and the implication (3) $\Longrightarrow$ (4) follows from Theorem 3.3, because $\widetilde{M}(x_0) = M(x_0)$. $\square$

Let us now see that the so-called (non-asymptotic) perturbation property of [10] holds in the case where D is described by finite-dimensional linear inequality constraints. Let X be a Hilbert space and C be a closed convex subset of X. Let $h_j \in X \setminus \{0\}$ $(j = 1, \ldots, m)$. Then by Riesz's Lemma, for each h_j, there exists a bounded linear functional f_j on X such that $f_j : X \longrightarrow \mathbb{R}$ is defined by

$$f_j(x) = \langle x, h_j \rangle \quad \forall\, x \in X;\ \ j = 1, \ldots, m.$$

Moreover, $\|h_j\| = \|f_j\| := \sup\{|f_j(x)| : x \in X,\ \ \|x\| = 1\}$ $(j = 1, \ldots, m)$. Let $Y := \mathbb{R}^m$, $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ and

$$S := \mathbb{R}^m_+ := \{z = (z_1, \ldots, z_m) \in \mathbb{R}^m : z_i \geqslant 0\ \forall\, i = 1, \ldots, m\}.$$

Define the function $g : X \longrightarrow \mathbb{R}^m$ by

$$g(x) := (f_1(x) - b_1, \ldots, f_m(x) - b_m) \quad \forall\, x \in X$$

and $D := \{x \in X : -g(x) \in S\}$. Then g is a continuous S-convex function and D is a closed convex subset of X. Define $K := C \cap D \neq \emptyset$. Let $g_i(x) := f_i(x) - b_i$ $(i = 1, \ldots, m;\ x \in X)$, and

$$I(x) := \{i \in \{1, 2, \ldots, m\} : g_i(x) = 0\} \quad \forall\, x \in K.$$

Note that if $x_0 \in K$ and $\lambda \in S^{\oplus}$, and if $(\lambda g)(x_0) = 0$, then $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, $\lambda_i \geqslant 0$ $(i = 1, \ldots, m)$ and $\lambda_i = 0$ for each $i \notin I(x_0)$. Note also that in the case of finitely many linear constraints, where $S = \mathbb{R}^m_+$ and $Y = \mathbb{R}^m$, the set $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is always a closed set in $X^* \times \mathbb{R}$. If $x_0 \in K$, then $\widetilde{M}(x_0) = M(x_0) = \cup_{\lambda \in S^{\oplus}} H_\lambda(x_0)$, where

$$H_\lambda(x_0) = \mathrm{cone}\left\{\sum_{i=1}^{m} \lambda_i h_i : \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+;\ \ \lambda_i = 0\ \forall\, i \notin I(x_0)\right\}.$$

To see this, first observe that if $\lambda \in S^{\oplus}$ and $(\lambda g)(x_0) = 0$, then $\partial(\lambda g)(x_0) = \left\{\sum_{i=1}^{m} \lambda_i h_i\right\}$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ for some $\lambda_i \geqslant 0$ $(i = 1, \ldots, m)$ and $\lambda_i = 0$ for each $i \notin I(x_0)$. This gives us that

$$M(x_0) = \cup_{\lambda \in S^{\oplus}} H_\lambda(x_0).$$

Since $\cup_{\lambda \in S^{\oplus}} \mathrm{Epi}\,(\lambda g)^*$ is closed in $X^* \times \mathbb{R}$, the result follows from Proposition 3.2.

By using Theorem 3.1, we now see that $\{C, D\}$ has the strong CHIP at $x_0 \in K$ is equivalent to the statement that, for any $x \in X$,

$$x_0 = P_K(x) \quad \text{if and only if} \quad x_0 = P_C\left(x - \sum_{i=1}^{m} \lambda_i h_i\right)$$

for some $\lambda_i \geqslant 0$ $(i = 1, \ldots, m)$ and $\lambda_i = 0$ for each $i \notin I(x_0)$.

4. Limiting dual conditions for best approximation

In this section we assume that X and Y are Banach spaces. We obtain limiting dual conditions characterizing best approximation over $K := C \cap D$ without a constraint qualification.

We first see how $(K-x)^{\ominus}$ can be expressed in terms of nets of ε-subgradients. We provide a proof which extends the proof of Lemma 3.1.

Lemma 4.1. *Let $x \in K$ be arbitrary. Then,*

$$(K-x)^{\ominus} = \left\{ \begin{array}{c} x^* \in X^* : \ x^* = w^* - \lim_{\alpha} (x^*_\alpha + y^*_\alpha), \ x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x), \ y^*_\alpha \in \partial_{r_\alpha}\delta_C(x), \\ \{\varepsilon_\alpha\} \text{ and } \{r_\alpha\} \subset \mathbb{R}_+, \{\lambda_\alpha\} \subset S^{\oplus}, \ \lim_{\alpha} (\lambda_\alpha g)(x) = 0, \\ \lim_{\alpha} \varepsilon_\alpha = 0 \text{ and } \lim_{\alpha} r_\alpha = 0. \end{array} \right\}.$$

Proof. Assume that $x^* \in (K-x)^{\ominus}$. Then, $\sigma_K(x^*) \leqslant x^*(x)$, and so $(x^*, x^*(x)) \in \text{Epi } \sigma_K$. Since (by (2.2))

$$\text{Epi } \sigma_K = w^* - \text{cl}\,(\cup_{\lambda \in S^{\oplus}} \text{Epi}\,(\lambda g)^* + \text{Epi}\,\sigma_C),$$

we have

$$(x^*, x^*(x)) \in w^* - \text{cl}\,(\cup_{\lambda \in S^{\oplus}} \text{Epi}\,(\lambda g)^* + \text{Epi}\,\sigma_C). \tag{4.1}$$

On the other hand,

$$\text{Epi}\,(\lambda g)^* = \cup_{\varepsilon \geqslant 0}\{(y^*, \varepsilon + y^*(x) - (\lambda g)(x)) : y^* \in \partial_\varepsilon(\lambda g)(x)\}$$

and

$$\text{Epi}\,\sigma_c = \text{Epi}\,\delta^*_C = \cup_{r \geqslant 0}\{(z^*, r + z^*(x) - \delta_C(x)) : z^* \in \partial_r \delta_C(x)\}.$$

Therefore, by (4.1) we have

$$\begin{aligned}(x^*, x^*(x)) \in w^* - \text{cl}\,(\cup_{\lambda \in S^{\oplus}} \cup_{\varepsilon \geqslant 0} \{(y^*, \varepsilon + y^*(x) \\ -(\lambda g)(x)) : y^* \in \partial_\varepsilon(\lambda g)(x)\} + \text{Epi}\,\delta^*_C).\end{aligned} \tag{4.2}$$

From (4.2), we can find nets $\{\varepsilon_\alpha\}, \{r_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^{\oplus}$ and $\{x^*_\alpha\}, \{y^*_\alpha\} \subset X^*$ with $x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ and $y^*_\alpha \in \partial_{r_\alpha}\delta_C(x)$ for all α, such that

$$(x^*, x^*(x)) = w^* - \lim_{\alpha} (x^*_\alpha + y^*_\alpha, \varepsilon_\alpha + r_\alpha + x^*_\alpha(x) + y^*_\alpha(x) - (\lambda_\alpha g)(x) - \delta_C(x)).$$

So,

$$x^* = w^* - \lim_{\alpha} (x^*_\alpha + y^*_\alpha) \tag{4.3}$$

and

$$x^*(x) = \lim_{\alpha} [\varepsilon_\alpha + r_\alpha + x^*_\alpha(x) + y^*_\alpha(x) - (\lambda_\alpha g)(x) - \delta_C(x)]. \tag{4.4}$$

In view of (4.3), (4.4) and that $x \in K \subset C$, we obtain

$$\lim_{\alpha} [x^*_\alpha(x) + y^*_\alpha(x)] = x^*(x) = \lim_{\alpha} [\varepsilon_\alpha + r_\alpha + x^*_\alpha(x) + y^*_\alpha(x) - (\lambda_\alpha g)(x)],$$

which implies that

$$\lim_{\alpha} [\varepsilon_\alpha + r_\alpha - (\lambda_\alpha g)(x)] = 0. \tag{4.5}$$

Since $x \in K \subset D$, then $-g(x) \in S$, and hence $-(\lambda_\alpha g)(x) \geqslant 0$ for all α. This, together with (4.5) and the fact that $\varepsilon_\alpha \geqslant 0$ and $r_\alpha \geqslant 0$ for all α, implies that $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = \lim_\alpha r_\alpha = 0$.

Conversely, suppose that there exist nets $\{\varepsilon_\alpha\}, \{r_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^\oplus$ and $\{x^*_\alpha\}, \{y^*_\alpha\} \subset X^*$ with $x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ and $y^*_\alpha \in \partial_{r_\alpha}\delta_C(x)$, for all α, such that $x^* = w^*-\lim_\alpha (x^*_\alpha + y^*_\alpha)$, and $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = \lim_\alpha r_\alpha = 0$. Since $x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ and $y^*_\alpha \in \partial_{r_\alpha}\delta_C(x)$ for all α, it follows that

$$(\lambda_\alpha g)(y) - (\lambda_\alpha g)(x) \geqslant x^*_\alpha(y - x) - \varepsilon_\alpha \quad \forall\, y \in X\ \forall\, \alpha, \tag{4.6}$$

and

$$\delta_C(y) \geqslant y^*_\alpha(y - x) - r_\alpha \quad \forall\, y \in X\ \forall\, \alpha. \tag{4.7}$$

If $y \in K \subset D$, then $-g(y) \in S$ and $\delta_C(y) = 0$. So, $(\lambda_\alpha g)(y) \leqslant 0$ for all α. Therefore, in view of (4.6) and (4.7), we obtain

$$(\lambda_\alpha g)(x) \leqslant (x^*_\alpha + y^*_\alpha)(x - y) + \varepsilon_\alpha + r_\alpha \quad \forall\, y \in K\ \forall\, \alpha. \tag{4.8}$$

Since $x^*(y) = \lim_\alpha (x^*_\alpha + y^*_\alpha)(y)$ for each $y \in X$, and $\lim_\alpha (\lambda_\alpha g)(x) = \lim_\alpha \varepsilon_\alpha = \lim_\alpha r_\alpha = 0$, it follows from (4.8) that for each $y \in K$, $x^*(y - x) \leqslant 0$. Hence, $x^* \in (K - x)^\ominus$. $\square$

Theorem 4.1. *Let X be a Hilbert space, $x \in X$, and let $x_0 \in K$. Then the following assertions are equivalent*:

(1) $x_0 = P_K(x)$;
(2) $x - x_0 = w^* - \lim_\alpha (x^*_\alpha + y^*_\alpha)$, $0 = \lim_\alpha (\lambda_\alpha g)(x_0)$, $0 = \lim_\alpha \varepsilon_\alpha$, $0 = \lim_\alpha r_\alpha$, *for some nets* $\{\varepsilon_\alpha\}$, $\{r_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^\oplus$ *and* $\{x^*_\alpha\}, \{y^*_\alpha\} \subset X^*$ *with* $x^*_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x_0)$ *and* $y^*_\alpha \in \partial_{r_\alpha}\delta_C(x_0)$ *for all* α.

Proof. By Lemma 3.2, we have $x_0 = P_K(x)$ if and only if $x - x_0 \in (K - x_0)^\ominus$. Hence, by Lemma 4.1, this is equivalent to (2). $\square$

Remark 4.1. Note that if X and Y are separable Banach spaces, then Lemmas 3.1, 4.1 and Theorem 4.1 hold for sequences, instead of nets.

The following example illustrates that in the absence of a constraint qualification the ε-subdifferentials in the description of the limiting dual conditions in Theorem 4.1 are essential.

Example 4.1. *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by*

$$g(x, y) = (x^2 + y^2)^{\frac{1}{2}} - y.$$

Let $x = (-1, 1) \in \mathbb{R}^2$, $S = \mathbb{R}_+$, and

$$C = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ 0 \leqslant y \leqslant 1\}.$$

Then C is a closed convex subset of $\mathbb{R}^2$, and

$$D = \{(x, y) \in \mathbb{R}^2 : -g(x, y) \in S\} = \{(x, y) \in \mathbb{R}^2 : x = 0,\ y \geqslant 0\}.$$

Now, let $x_0 = (0, 1) \in K := C \cap D$. It is clear that $x_0 = P_K(x)$. Also, observe that for any $\lambda > 0$ and $\varepsilon > 0$, we have

$$\partial_\varepsilon(\lambda g)(x_0) = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + \lambda)^2 \leqslant \lambda^2,\ y \geqslant -\varepsilon\}.$$

For each $n = 1, 2, \ldots$, let $\varepsilon_n = \frac{1}{n}$, $r_n = 0$, $\lambda_n = \frac{1}{2}(n + \frac{2}{n}) + 1$, $x_n^* = (-1 - \frac{1}{n}, -\frac{1}{n})$, and $y_n^* = 0$. Then, $x_n^* \in \partial_{\varepsilon_n}(\lambda_n g)(x_0)$, $y_n^* \in \partial_{r_n}\delta_C(x_0)$, $(\lambda_n g)(x_0) = 0$ for all $n \geqslant 1$, $\lim_{n\to+\infty} \varepsilon_n = \lim_{n\to+\infty} r_n = 0$, and

$$w^* - \lim_{n\to+\infty} (x_n^* + y_n^*) = (-1, 0) = x - x_0.$$

Note that $(-1, 0) \notin cl(M(x_0)$, since, for each $\lambda \in S^\oplus$, $(\lambda g)(x_0) = 0$ and $\partial(\lambda g)(x_0) = \{(0, 0)\}$.

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